Mixed effects models

DAAG Chapter 10

Learning objectives

In this section, we will learn about mixed effects models (also known as multilevel modelling).

- Random effects
  - What are random effects? How do they differ from fixed effects?
  - How can we include random effects in the linear model framework?
- Multilevel modelling with mixed effects
  - Getting the error structure right is critical
  - Complete pooling, no pooling, and partial pooling
  - Including predictors
  - Prediction depends on the population of interest

Mixed effects models: motivating problem

Survey data on student attitude towards science.

- 1385 students
- 20 classes in 12 private schools
- 46 classes in 29 public schools
- Data are scores from 1 (dislike) to 12 (like)
- The number of students in each class is different (range: 3 to 50)

Of interest:

- Difference between private and public schools
- Difference between girls and boys
- Are there differences between schools, and classes within schools, greater than would be due to differences between students?

Mixed effects models: motivating problem

What are the sources of variation in this data?

- Sex effect
- School type effect

... but also ...

- School effect
- Class effect
- Student effect

Some of these effects are *fixed effects*, and some are *random effects.*
Notice also that some of these effects act at different scales – there are groupings in the data, and the groups are nested (student within class, class within school).

Fixed effects, random effects

Characteristics of fixed effects:

- Inference is limited to the levels observed
- In a designed experiment, the levels are chosen by the experimenter
- Examples (from previous): Sex effect and School type (private vs public)

Characteristics of random effects:

- Inference can be generalized to other levels that were not observed
- In a designed experiment, the levels were chosen randomly
- There is nothing "special" about the levels included
- Examples (from previous): School effect and Class effect and Student effect

Putting mixed effects into the linear model framework

Using the science attitudes data, we wish to model

$$\text{Attitude} = \text{sex} + \text{type} + \text{school} + \text{class} + \text{student}$$

Here we will suppose that the fixed effects *sex* and *type* act at the student level. Using the familiar linear model framework,

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \alpha_{1j[i]} + \alpha_{2k[i]} + \epsilon_i$$

where

- y_i is the attitude score for student i
- β_0 is the overall intercept
- β_1 is the coefficient for the sex effect, x_{1i} is the sex of student i
- β_2 is the coefficient for the type effect, x_{2i} is the school type of student i
- $\alpha_{1j[i]}$ is the random effect of school j corresponding to student i
- $\alpha_{2k[i]}$ is the random effect of class k corresponding to student i

Putting mixed effects into the linear model framework

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The vectors of random effects α_1, α_2, and ϵ each have their own distribution, i.e.

- $\alpha_{1j} \sim N(0, \sigma_1^2)$ for all schools j
- $\alpha_{2k} \sim N(0, \sigma_2^2)$ for all classes k
- $\epsilon_i \sim N(0, \sigma^2)$ for all students i

Science attitude fit

```
Random effects:
 Groups       Name        Variance Std.Dev.
 school:class (Intercept) 0.3206   0.5662
 school       (Intercept) 0.0000   0.0000
 Residual                 3.0521   1.7470
Number of obs: 1383, groups: school:class, 66; school, 41

Fixed effects:
              Estimate Std. Error t value
(Intercept)     4.7218     0.1624  29.071
sexm            0.1823     0.0982   1.857
PrivPubpublic   0.4117     0.1857   2.217

Correlation of Fixed Effects:
            (Intr) sexm
sexm        -0.309
PrivPubpblc -0.795  0.012
```

Science attitude fit - no school effects

```
Random effects:
 Groups       Name        Variance Std.Dev.
 school:class (Intercept) 0.3206   0.5662
 Residual                 3.0521   1.7470
Number of obs: 1383, groups: school:class, 66

Fixed effects:
              Estimate Std. Error t value
(Intercept)     4.7218     0.1624  29.072
sexm            0.1823     0.0982   1.857
PrivPubpublic   0.4117     0.1857   2.217

Correlation of Fixed Effects:
            (Intr) sexm
sexm        -0.309
PrivPubpblc -0.795  0.012
```

Science attitude fit

What have we learned?

- The best estimate of the sex effect is 0.1823 points higher for males
- The best estimate of the school type effect is 0.4177 points higher for public
  - Both of these fixed effects are marginally significant
- The proportion of variation due to differences between schools (aside from public-private effect) is approximately zero
- The proportion of variation due to differences between classes is $0.321/(0.321 + 3.05) = 9.5\%$
- The proportion of variation due to differences between students is $3.05/(0.321 + 3.05) = 91.5\%$

Getting the error structure wrong: ignoring class effects

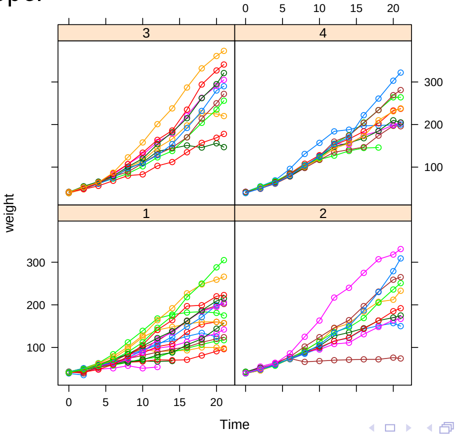
```
Random effects:
 Groups   Name        Variance Std.Dev.
 school   (Intercept) 0.1655   0.4068
 Residual             3.2185   1.7940
Number of obs: 1383, groups: school, 41

Fixed effects:
              Estimate Std. Error t value
(Intercept)     4.7377     0.1634  29.001
sexm            0.1969     0.1007   1.956
PrivPubpublic   0.4168     0.1852   2.250

Correlation of Fixed Effects:
            (Intr) sexm
sexm        -0.274
PrivPubpblc -0.807 -0.031
```

Getting the error structure wrong: ignoring class and school effects

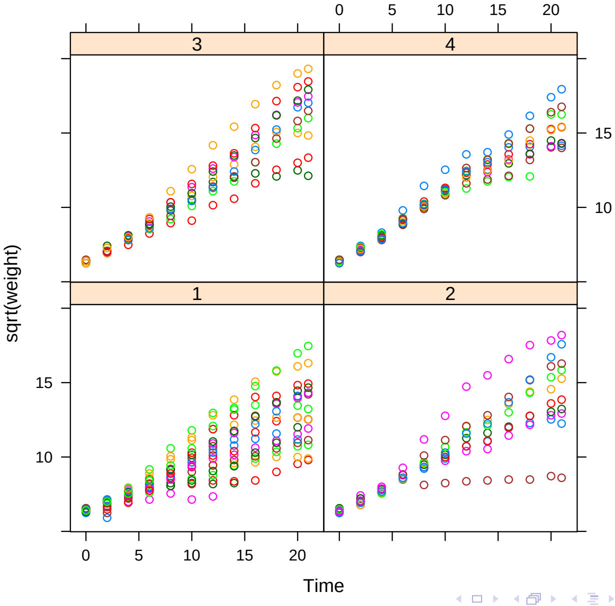
```
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)    4.74024    0.09955  47.616  < 2e-16 ***
sexm           0.15093    0.09860   1.531 0.126064
PrivPubpublic  0.39507    0.10511   3.759 0.000178 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.833 on 1380 degrees of freedom
  (2 observations deleted due to missingness)
Multiple R-squared:  0.01175,   Adjusted R-squared:  0.01032
F-statistic: 8.203 on 2 and 1380 DF,  p-value: 0.0002873
```

Random coefficients: Random slopes

- Random effects can enter as random slopes.
- Include a fixed effect for the average slope, plus random effects for adjustments to the slope.
- Chick weights over time for chicks on four different diets.
- Model with random slopes for each chick, and a fixed effect of diet on slope.



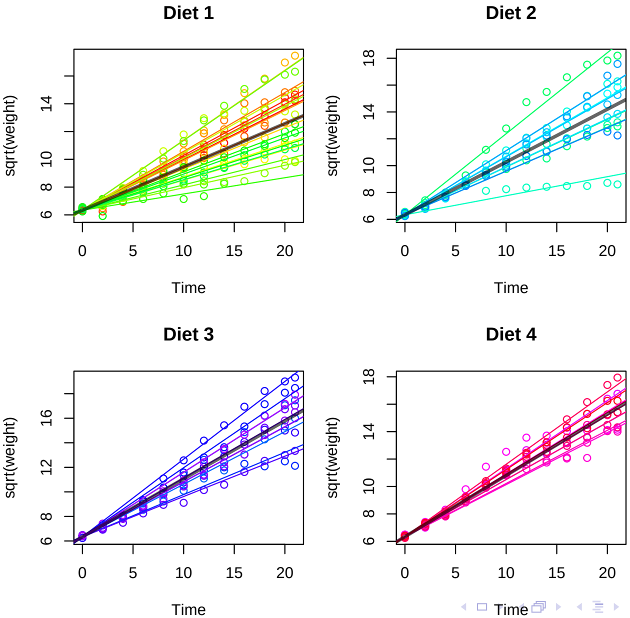
Chicks - square root transformation



Chicks - mixed effects model fit

```
Formula: sqrt(weight) ~ Time + (Time - 1 | Chick) + Diet:Time
   Data: ChickWeight

Random effects:
 Groups   Name Variance Std.Dev.
 Chick    Time 0.01043  0.1021
 Residual      0.23868  0.4886
Number of obs: 578, groups: Chick, 50

Fixed effects:
            Estimate Std. Error t value
(Intercept)  6.33795    0.03833  165.35
Time         0.31156    0.02369   13.15
Time:Diet2   0.08167    0.04010    2.04
Time:Diet3   0.16085    0.04010    4.01
Time:Diet4   0.13527    0.04011    3.37
```

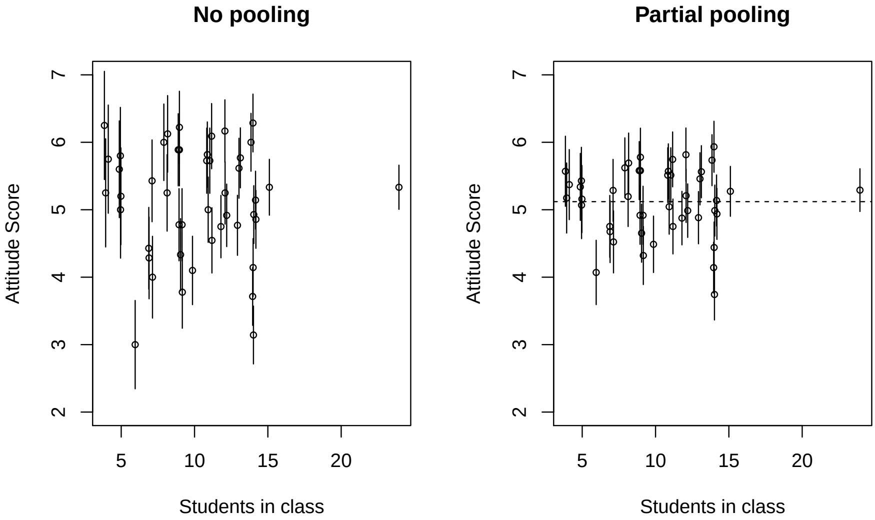
Chicks - mixed effects model fit



Pooling

- Pooling refers to the aggregation of observations into groups.
- Going back to our initial example, suppose we want to calculate average attitude scores:
  - *No pooling:* In this case, we treat each classroom as a replicate
  - *Complete pooling:* In this case, we pool all classrooms together
  - *Partial pooling:* A compromise achieved by modelling classroom effects as a group (mixed effects model)

Pooling



No pooling
Attitude Score
2
3
4
5
6
7
5
10
15
20
Students in class
Partial pooling
Attitude Score
2
3
4
5
6
7
5
10
15
20
Students in class